Day 5: Everything Else

Introduction to Multilevel Models
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Generalized linear model specification

As in models for continuous responses, we are interested in the expectation (mean) of the response as a function of the covariate. The expectation of a binary (0 or 1) response is just the probability that the response is 1:

$$E(y_i|x_i) = \Pr(y_i = 1|x_i)$$

In linear regression, the conditional expectation of the response is modeled as a linear function $E(y_i|x_i) = \beta_1 + \beta_2 x_i$ of the covariate (see sec. 1.5). For dichotomous responses, this approach may be problematic because the probability must lie between 0 and 1, whereas regression lines increase (or decrease) indefinitely as the covariate increases (or decreases). Instead, a nonlinear function is specified in one of two ways:

$$\Pr(y_i = 1|x_i) = h(\beta_1 + \beta_2 x_i)$$

or

$$g\{\Pr(y_i = 1|x_i)\} = \beta_1 + \beta_2 x_i = \nu_i$$

where ν_i is referred to as the *linear predictor*. These two formulations are equivalent if the function $h(\cdot)$ is the inverse of the function $g(\cdot)$. Here $g(\cdot)$ is known as the *link function* and $h(\cdot)$ as the *inverse link function*, sometimes written as $g^{-1}(\cdot)$.

Link function for binary response

$$\Pr(y_i = 1|x_i) = \text{logit}^{-1}(\beta_1 + \beta_2 x_i) \equiv \frac{\exp(\beta_1 + \beta_2 x_i)}{1 + \exp(\beta_1 + \beta_2 x_i)}$$

or

$$\text{logit}\left\{\Pr(y_i = 1|x_i)\right\} \equiv \ln \underbrace{\left\{ \frac{\Pr(y_i = 1|x_i)}{1 - \Pr(y_i = 1|x_i)} \right\}}_{\text{Odds}_{(y_i=1|x_i)}} = \beta_1 + \beta_2 x_i$$

Logit models as latent response models

The logistic regression model and other models for dichotomous responses can also be viewed as latent-response models. Underlying the observed dichotomous response y_i (whether the woman works or not), there is an unobserved or latent continuous response y_i^*, representing the propensity to work or the excess utility of working as compared with not working. If this latent response is greater than 0, the observed response is 1:

$$y_i = \begin{cases} 1 & \text{if } y_i^* > 0 \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, we will assume that there is one covariate x_i. A linear regression model is then specified for the latent response y_i^*

$$y_i^* = \beta_1 + \beta_2 x_i + \epsilon_i$$

where ϵ_i is a residual error term with $E(\epsilon_i|x_i) = 0$ and the error terms of different women i are independent.

"logit" regression v. "progit" regression

- Logit regression:

In logistic regression, ϵ_i is assumed to have a logistic cumulative density function given x_i,

$$\Pr(\epsilon_i < \tau | x_i) = \frac{\exp(\tau)}{1 + \exp(\tau)}$$

which has mean zero and variance $\pi^2/3 \approx 3.29$ (note that π here represents the famous mathematical constant 'pi').

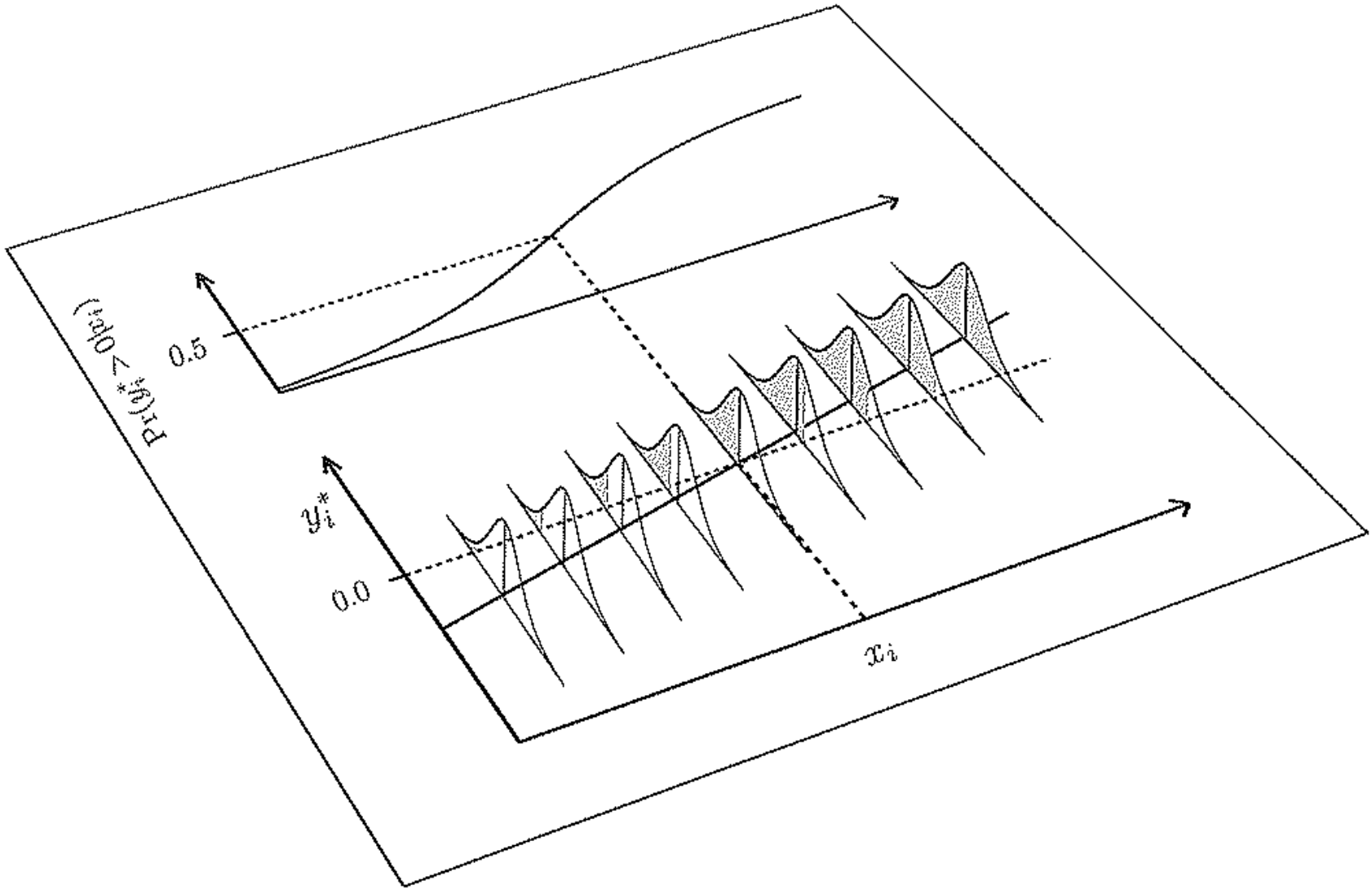
- Probit regression:

When a latent-response formulation is used, it seems natural to assume that ϵ_i has a normal distribution given x_i, as is usually done in linear regression. If a standard (mean 0 and variance 1) normal distribution is assumed, the model becomes a probit model

$$\begin{aligned} \Pr(y_i = 1 | x_i) &= \Pr(y_i^* > 0 | x_i) = \Pr(\beta_1 + \beta_2 x_i + \epsilon_i > 0) \\ &= \Pr\{\epsilon_i > -(\beta_1 + \beta_2 x_i)\} = \Pr(-\epsilon_i \leq \beta_1 + \beta_2 x_i) \\ &= \Pr(\epsilon_i \leq \beta_1 + \beta_2 x_i) = \Phi(\beta_1 + \beta_2 x_i) \end{aligned} \tag{6.4}$$

Here $\Phi(\cdot)$ is the standard normal cumulative distribution function, the probability that a standard normally distributed random variable (here ϵ_i) is less than the argument. $\Phi(\cdot)$ is the inverse link function $h(\cdot)$, whereas the link function $g(\cdot)$ is $\Phi^{-1}(\cdot)$, the inverse standard normal cumulative distribution function, called the *probit link* function. The penultimate equality in (6.4) exploits the symmetry of the normal distribution.

Illustration of logit GLM and latent response formulations



Random-intercept logistic regression

- To relax the assumption of conditional independence we add a group-specific random intercept ζ_j to the linear predictor:

$$\text{logit}\{\Pr(y_{ij}=1|\mathbf{x}_{ij},\zeta_j)\} = \beta_1 + \beta_2 x_{2j} + \beta_3 x_{3ij} + \beta_4 x_{2j} x_{3ij} + \zeta_j$$

- We assume that $y_{ij}|\pi_{ij} \sim \text{binomial}(1, \pi_{ij})$, given that $\pi ij \equiv \Pr(y_{ij}|\mathbf{x}_{ij}, \zeta_j)$
- We can estimate this model using `xtlogit`
- Alternatively we can use the `xtmelogit` command, but we must specify the number of integration points

Poisson models

- This is also a GLM, but with different link and error functions
- Focus here is on a constant incidence rate λ defined as the instantaneous probability of a new event per time interval
- The number of events y that occur in time t has a Poisson distribution:

$$\Pr(y|\mu) = \frac{\exp(-\mu)\mu^y}{y!}$$

where $\mathrm{E}(y) = \mu$ and is given by

$$\mu = \lambda t$$

Specification for single-level Poisson regression

The expected number of visits μ_{ij} at occasion i for subject j is specifie
linear model

$$\ln(\mu_{ij}) = \nu_{ij} = \beta_1 + \beta_2 x_{2i} + \cdots + \beta_7 x_{7ij}$$

or equivalently as an exponential model for the expected number of visits:

$$\mu_{ij} = \exp(\nu_{ij})$$

Random intercept model for multilevel Poisson regression

One way to address the dependence within persons is to include a person-spec intercept ζ_{1j} in the Poisson regression model

$$\begin{aligned}\mu_{ij} \equiv E(y_{ij}|\mathbf{x}_{ij}, \zeta_{1j}) &= \exp(\beta_1 + \beta_2 x_{2i} + \cdots + \beta_7 x_{7ij} + \zeta_{1j}) \\ &= \exp\{(\beta_1 + \zeta_{1j}) + \beta_2 x_{2i} + \cdots + \beta_7 x_{7ij}\} \\ &= \exp(\zeta_{1j}) \exp(\beta_1 + \beta_2 x_{2i} + \cdots + \beta_7 x_{7ij})\end{aligned}$$

where $\zeta_{1j}|\mathbf{x}_{ij} \sim \mathrm{N}(0, \psi_{11})$ and the ζ_{1j} are independent across persons j. The of the random intercept, $\exp(\zeta_{1j})$, is sometimes called a *frailty*. The number

- Commands are `xtpoisson` and `xtmepoisson`
- It is also possible to run random coefficients models for Poisson